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ON GENERALIZED MOMENTUM

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Abstract—We point out that, with a simple modification of reasoning commonly used, one can get, easily, thermodynamic restrictions on constitutive equations for generalized momenta.

1. INTRODUCTION

Various theories of structured continua involve constitutive equations for generalized momenta. Some, but not all writers assume these to depend linearly on generalized velocities. For example, Cowin and Leslie[1] consider more general possibilities for Cosserat continua, using considerations of invariance to obtain some restrictions on the form of such constitutive equations. Similarly, Hills and Roberts[2] introduce more general forms, in their theory of superfluids. They employ ideas of invariance, and the Clausius–Duhem inequality, to get restrictions on the various constitutive equations considered. In dealing with theories of structured continua, I have got into the habit of using a line of thought, not used by them, which makes it rather routine to get thermodynamic restrictions on constitutive equations for generalized momenta. Also, it played a role in developing the general transformation theory for various continuum theories presented by Ericksen[3]. My purpose is to elaborate this.

Typical continuum theories involve a list of equations of a common form. The reader might find it easier to recognize the form to be given, if he thinks in terms of three-dimensional theories, employing the time t and rectangular Cartesian material or spatial coordinates x^i ($i = 1, 2, 3$) as independent variables. As a personal matter, I use the same format when using curvilinear coordinates, or when dealing with mechanics of surfaces, etc. The form is

$$\frac{\partial P_\alpha}{\partial t} = T^i_{\alpha,i} + F^I_\alpha + F^E_\alpha, \quad \alpha = 1, \dots, n, \tag{1}$$

with P_α interpreted as density of (generalized) momenta, T^i_α as a kind of generalized stress, F^I_α as an internal body force, all things for which constitutive equations are needed. Here, F^E_α covers external body forces of external origin, like the gravitational effect produced by another massive body, not to be given by a definite constitutive equation. Commonly, we split these equations into subsets, each subset being identified with components of some tensor. How we do this depends on which special kind of theory we are considering, so I won't belabor this. In the work of Cowin and Leslie[1], for example, these take the form of a vector equation, and a second-order tensor equation. Also, if we are using spatial coordinates, T^i_α will include analogs of Reynold's stresses, described more explicitly below.

To put these in a different form, more like what is likely to be seen in some of the literature, we need another equation, covering conservation of mass. A representative form is

$$\frac{\partial \rho}{\partial t} + (\rho v^i), \quad i = 0, \tag{2}$$

where ρ is the mass density, $v^i = 0$ if we are using material coordinates, and v^i are the usual velocity components, if we are using spatial coordinates. For simplicity, we exclude mixture theories, which involve additional mass balances, covering chemical reactions between different constituents; and other complications. The theory of Hills and Roberts[2] is a kind of mixture theory, so our format will not quite fit their theory. Then, we can introduce the material derivative, for any function f, viz.

$$\dot{f} = \frac{\partial f}{\partial t} + f_{,i}\, v^i, \tag{3}$$

write

$$P_\alpha = \rho p_\alpha, \tag{4}$$

and juggle (1) into the form

$$\rho \dot{p}_\alpha = t^i_{\alpha,i} + F^I_\alpha + F^E_\alpha. \tag{5}$$

Here, t^i_α is more like what we commonly consider to be the stress tensor, it and T^i_α being related by

$$T^i_\alpha = t^i_\alpha - P_\alpha v^i = t^i_\alpha - \rho p_\alpha v^i, \tag{6}$$

the difference being the analog of Reynolds' stress. Granted (2), (1) and (5) are equivalent, so one can use whichever seems most convenient.

To make any use of thermodynamics, one needs an energy equation. A typical form of this is

$$\frac{\partial E}{\partial t} = E^i{}_{,i} + F, \tag{7}$$

where E is the (total) energy density, E^i is the energy flux, and F covers any volume sources. In some of the more general theories, the decomposition of E into an internal energy and kinetic energy is a somewhat ambiguous matter and, I think, it is better to get out of the habit. Generally, we detail (7) in a form suggested by the first law of thermodynamics, writing

$$\left.\begin{aligned} E &= \rho e, \\ E^i &= -\rho e v^i + t^i_\alpha w^\alpha - Q^i, \\ F &= F^E_\alpha w^\alpha - R, \end{aligned}\right\} \tag{8}$$

using the summation convention for Greek as well as Latin indices. Here, the w^α represent generalized velocities. Generally, v^i will be included among the w^α or some simple *a priori* relation will be assumed to relate them, so v^i and w^α are not really considered as independent variables. Also, Q^i is interpreted as the heat flux, and R represents radiation. Then, using (2) and (5), we use (8) to reduce (7) to the form

$$\rho \dot{e} = t^i_\alpha w^\alpha{}_{,i} + \rho(\dot{p}_\alpha - F^I_\alpha)\, w^\alpha - Q^i{}_{,i} - R. \tag{9}$$

Another form is suggested by the transformation theory described by Ericksen[3]. In (8), we can use (6) to replace t^i_α by T^i_α, which gives

$$E^i = \rho a v^i + T^i_\alpha w^\alpha - Q^i, \tag{10}$$

where

$$a = p_\alpha w^\alpha - e \tag{11}$$

is interpretable as action per unit mass. In a rather natural way, this brings action into the picture. Following the lead, we generate an equation for a, which is easily found to be

$$\rho \dot{a} + T^i_\alpha w^\alpha{}_{,i} - P_\alpha \frac{\partial w^\alpha}{\partial t} - F^I_\alpha\, w^\alpha = Q^i{}_{,i} + R, \tag{12}$$

Alternatively, one can use (11) to rearrange (9) into an equivalent form, viz.

$$\rho \dot{a} + t^i_\alpha w^\alpha{}_{,i} - \rho p_\alpha \dot{w}^\alpha - F^I_\alpha w^\alpha = Q^i{}_{,i} + R, \tag{13}$$

but it seemed worthwhile to mention one of the things which motivated me to introduce action.

Commonly, we will also appeal to some version of the second law. For simplicity, I will use the Clausius–Duhem inequality, a rather common choice, viz.

$$\rho \dot{\eta} + (Q^i/\theta)_{,i} + R/\theta \geq 0, \tag{14}$$

where η is entropy per unit mass, and $\theta > 0$ denotes absolute temperature. Following a line of thought which has become rather standard we then set

$$b = a + \theta\eta, \tag{15}$$

a substitution not entirely unlike (11), reducing (13) to the inequality

$$\rho \dot{b} + t^i_\alpha w^\alpha{}_{,i} - \rho p_\alpha \dot{w}^\alpha - F^I_\alpha w^\alpha - \rho\eta\dot{\theta} \geqq Q^i\theta_{,i}/\theta. \tag{16}$$

Any definite proposal concerning treatment of generalized momenta must be regarded as speculative, but one line of thought seems to me sensible. With (16), it is natural to consider giving constitutive equations for b, t^i_α, p_β, *etc.* requiring that these be consistent with (16) for all processes. With (11) and (15), we will get, indirectly, a constitutive equation for e. With some simpler kinds of constitutive assumptions, (16) leads to equations of the form

$$p_\alpha = \partial b/\partial w^\alpha, \quad \eta = \partial b/\partial \theta. \tag{17}$$

Then, a becomes a Legendre transform of b, e a Legendre transform of a, in accord with experience for conservative systems. The various writers who have been willing to assume (17), have, in effect, cancelled out some terms in (16). Cowin and Leslie[1] did not, so one can use (16) to get additional restrictions on their constitutive equations for momenta, for example. With (16), there is the suggestion that, sometimes, part of p_α might contribute to dissipation. I have not thought of a compelling reason to exclude the possibility, or an argument making plausible that the possibility is real.

Clearly, the idea of replacing energy by action can be applied to theories which don't quite fit the above format, such as those involving generalizations of the Clausius–Duhem inequality of the kind preferred by Müller[4]. Some adjustments of the format are required, to include mixture theories.

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